

Einstein and Møller energy-momentum complexes for a new regular black hole solution with a nonlinear electrodynamics source

I. Radinschi*

Department of Physics , Gh. Asachi Technical University, Iasi, 700050, Romania

Farook Rahaman†

Department of Mathematics, Jadavpur University, Kolkata 700 032, West Bengal, India

Th. Grammenos‡

Department of Civil Engineering, University of Thessaly, 383 34 Volos, Greece

Sayeedul Islam§

Department of Mathematics, Jadavpur University, Kolkata 700 032, West Bengal, India.

(Dated: October 3, 2016)

A study about the energy and momentum distributions of a new charged regular black hole solution with a nonlinear electrodynamics source is presented. The energy and momentum are calculated using the Einstein and Møller energy-momentum complexes. The results show that in both pseudotensorial prescriptions the expressions for the energy of the gravitational background depend on the mass M and the charge q of the black hole, an additional factor β coming from the spacetime metric considered, and the radial coordinate r , while in both prescriptions all the momenta vanish. Further, it is pointed out that in some limiting and particular cases the two complexes yield the same expression for the energy distribution as that obtained in the relevant literature for the Schwarzschild black hole solution.

* radinschi@yahoo.com

† rahaman@associates.iucaa.in

‡ thgramme@uth.gr

§ sayeedul.jumath@gmail.com

I. INTRODUCTION

Energy-momentum localization plays a leading role in the theories advanced over the years in relation to General Relativity. There is a major difficulty, however, in formulating a proper definition for the energy density of gravitational backgrounds. Indeed, the key problem is the lack of a satisfactory description for the gravitational energy.

Many researchers have conducted extensive research using different methods for energy-momentum localization. Standard research methods include the use of different tools, such as super-energy tensors [1], quasi-local expressions [2] and the famous energy-momentum complexes of Einstein [3], Landau-Lifshitz [4], Papapetrou [5], Bergmann-Thomson [6], Møller [7], Weinberg [8], and Qadir-Sharif [9]. The main problem encountered is the dependence on the reference frame of these pseudotensorial prescriptions. An alternative method used in many studies on computing the energy and momentum distributions in order to avoid the dependence on coordinates is the teleparallel theory of gravitation [10].

As regards pseudotensorial prescriptions, only the Møller energy-momentum complex is a coordinate independent tool. Schwarzschild Cartesian coordinates and Kerr-Schild Cartesian coordinates are useful to compute the energy-momentum in the case of the other pseudotensorial definitions. Over the past few decades, despite the criticism directed against energy-momentum complexes concerning mainly the physicalness of the results obtained by them, their application has provided physically reasonable results for many spacetime geometries, more particularly for geometries in $(3+1)$, $(2+1)$ and $(1+1)$ dimensions [11]-[12].

There is an agreement between the Einstein, Landau-Lifshitz, Papapetrou, Bergmann-Thomson, Weinberg and Møller prescriptions, on the one hand, and the definition of the quasi-local mass advanced by Penrose [13] and developed by Tod [14] for some gravitating systems, on the other hand (see [15] for a comprehensive review). Several pseudotensorial definitions “provide the same results” for any metric of the Kerr-Schild class and for solutions that are more general than the Kerr-Schild class (see, for example, the works of J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, and S. S. Xulu in [11], and K. S. Virbhadra in [16]). Furthermore, the similarity between some of the aforementioned results and those obtained by using the teleparallel theory of gravitation [17] cannot be overlooked. In fact, the history of energy-momentum complexes should include their definition and use, as well as the attempts for their rehabilitation [18].

The present work has the following structure: in Section 2 we describe the new spherically symmetric, static, charged regular black hole solution with a nonlinear electrodynamics source [19] under study. Section 3 is focused on the presentation of the Einstein and Møller energy-momentum complexes used for performing the calculations. Section 4 contains the calculations of the energy and momentum distributions. In the Discussion and Final Remarks given in Section 5, we make a brief description of the results of our investigation as well as some limiting and particular cases. Throughout the article we use geometrized units ($c = G = 1$), the signature chosen for our purpose is $(+, -, -, -)$, and the calculations are performed using the Schwarzschild Cartesian coordinates $\{t, x, y, z\}$ for the Einstein prescription and the Schwarzschild coordinates $\{t, r, \theta, \varphi\}$ for the Møller prescription. Also, Greek indices range from 0 to 3, while Latin indices run from 1 to 3.

II. DESCRIPTION OF THE NEW REGULAR BLACK HOLE SOLUTION WITH A NONLINEAR ELECTRODYNAMICS SOURCE

In this section we present a new spherically symmetric, static, charged regular black hole solution with a nonlinear electrodynamics source recently developed by L. Balart and E. C. Vagenas [19] and we analyze the distribution of energy and momentum using the Einstein and Møller prescriptions.

A brief but interesting discussion about the regular black hole solutions that have been obtained by coupling gravity to nonlinear electrodynamics theories is presented in [19] (see Introduction and References therein for more details). Further, in an interesting and similar work the horizon entropy of a black hole is determined as a function of Komar energy and the horizon area [20].

In order to develop the new charged regular black hole solution, the authors of [19] considered the Fermi-Dirac-type distribution. For this purpose, they generalized the methodology developed in Sec. 2 of their paper by considering distribution functions raised to the power of a real number greater than zero. We notice that the methodology presented in Sec. 2 consists in constructing a general charged regular black hole metric for mass distribution functions that are inspired by continuous probability density distributions. The corresponding electric field for each black hole solution is also constructed in terms of a general mass distribution function. The metric function is given by

$$f(r) = 1 - \frac{2M}{r} \left(\frac{\sigma(\beta r)}{\sigma_\infty} \right)^\beta, \quad (1)$$

where $\sigma_\infty = \sigma(r \rightarrow \infty)$ is a normalization factor and the function $\sigma(\beta r)$ corresponds to any one of the mass functions listed in Table 1 of [19], but with the coordinate r multiplied by an additional factor $\beta > 0$.

The new spherically symmetric, static, charged regular black hole solution with a nonlinear electrodynamics source given by eq. (29) in [19] is obtained, as we pointed out above, using the Fermi-Dirac-type distribution, and the metric function becomes now

$$f(r) = 1 - \frac{2M}{r} \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta. \quad (2)$$

Moreover, when $r \rightarrow \infty$ the mass function $m(r) = M\left(\frac{\sigma(\beta r)}{\sigma_\infty}\right)^\beta \rightarrow M$. The distribution function satisfies the condition $\frac{\sigma(r)}{\sigma_\infty} \rightarrow 1$ when $r \rightarrow \infty$. This solution is a generalization of the Ayón-Beato and García black hole solution [21].

The corresponding electric field has the expression

$$E(r) = \frac{q}{r^2} \exp\left(\frac{(1-\beta)q^2}{2\beta M r}\right) \left[\operatorname{sech}\left(\frac{q^2}{2\beta M r}\right) \right]^{1+\beta} \times \left[1 - \frac{q^2}{4Mr} \tanh\left(\frac{q^2}{2\beta M r}\right) + \frac{1}{4\beta M r} \left(\frac{1-\beta}{\exp\left\{\frac{q^2}{\beta M r}\right\} + 1} \right) \right]. \quad (3)$$

In order to construct the extremal regular black hole metric for this example, some values of β and the corresponding charges are listed in Table 2 of [19].

Finally, the new charged regular black hole solution with a nonlinear electrodynamics source is described by the metric

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (4)$$

with $B(r) = f(r)$, $A(r) = \frac{1}{f(r)}$.

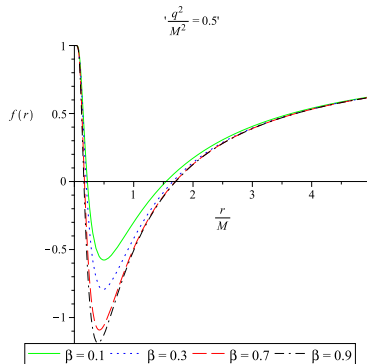


FIG. 1. An inner and an outer horizon exist at the points where $f(r)$ meets the $\frac{r}{M}$ -axis, here shown for four different values of the parameter β .

Figure 1 shows that two horizons exist at the points where $f(r)$ meets the $\frac{r}{M}$ -axis and for four different values of the parameter β . We have chosen $\left(\frac{q}{M}\right)^2 = 0.5$. Note that the positions of the inner and the outer horizon remain unaffected for various values of β .

III. EINSTEIN AND MÖLLER ENERGY-MOMENTUM COMPLEXES

The Einstein energy-momentum complex [3] for a $(3+1)$ dimensional gravitational background has the well-known expression

$$\theta_\nu^\mu = \frac{1}{16\pi} h_{\nu,\lambda}^{\mu\lambda}. \quad (5)$$

The superpotentials $h_\nu^{\mu\lambda}$ involved in (5) are given by

$$h_\nu^{\mu\lambda} = \frac{1}{\sqrt{-g}} g_{\nu\sigma} [-g(g^{\mu\sigma} g^{\lambda\kappa} - g^{\lambda\sigma} g^{\mu\kappa})]_{,\kappa} \quad (6)$$

which satisfies the necessary antisymmetric property:

$$h_\nu^{\mu\lambda} = -h_\nu^{\lambda\mu}. \quad (7)$$

In the Einstein prescription the local conservation law is respected

$$\theta_{\nu,\mu}^\mu = 0. \quad (8)$$

Thus, the energy and momentum can be evaluated in Einstein's prescription with

$$P_\mu = \int \int \int \theta_\mu^0 dx^1 dx^2 dx^3. \quad (9)$$

Here, θ_0^0 and θ_i^0 represent the energy and momentum density components, respectively.

Applying Gauss' theorem the energy-momentum reads

$$P_\mu = \frac{1}{16\pi} \int \int h_\mu^{0i} n_i dS, \quad (10)$$

with n_i the outward unit normal vector over the surface dS . In eq. (10) P_0 is the energy. Concerning the expression for the Møller energy-momentum complex [7] we have

$$\mathcal{J}_\nu^\mu = \frac{1}{8\pi} M_{\nu,\lambda}^{\mu\lambda}, \quad (11)$$

with the Møller superpotentials $M_\nu^{\mu\lambda}$ given by

$$M_\nu^{\mu\lambda} = \sqrt{-g} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma}. \quad (12)$$

The Møller superpotentials $M_\nu^{\mu\lambda}$ are antisymmetric:

$$M_\nu^{\mu\lambda} = -M_\nu^{\lambda\mu}. \quad (13)$$

Like the Einstein energy-momentum complex, Møller's energy-momentum complex also satisfies the local conservation law

$$\frac{\partial \mathcal{J}_\nu^\mu}{\partial x^\mu} = 0. \quad (14)$$

In (14) \mathcal{J}_0^0 gives the energy density and \mathcal{J}_i^0 represents the momentum density components.

For the Møller prescription, the energy and momentum distributions are obtained by

$$P_\mu = \int \int \int \mathcal{J}_\mu^0 dx^1 dx^2 dx^3 \quad (15)$$

and the energy distribution can be calculated by

$$E = \int \int \int \mathcal{J}_0^0 dx^1 dx^2 dx^3. \quad (16)$$

Again, using Gauss' theorem one gets

$$P_\mu = \frac{1}{8\pi} \int \int M_\mu^{0i} n_i dS. \quad (17)$$

IV. ENERGY AND MOMENTUM DISTRIBUTION FOR THE NEW REGULAR BLACK HOLE SOLUTION WITH A NONLINEAR ELECTRODYNAMICS SOURCE

In order to compute the energy and momenta in the Einstein prescription, it is useful to transform the metric given by the line element (4) in Schwarzschild Cartesian coordinates applying the coordinate transformation $x = r \sin \theta \cos \varphi$, $y = r \sin \theta \sin \varphi$, $z = r \cos \theta$. Then, the following line element is obtained:

$$ds^2 = B(r)dt^2 - (dx^2 + dy^2 + dz^2) - \frac{A(r) - 1}{r^2}(xdx + ydy + zdz)^2. \quad (18)$$

The components of the superpotential h_μ^{0i} in quasi-Cartesian coordinates for $\mu = 1, 2, 3$ and $i = 1, 2, 3$ are given by

$$h_1^{01} = h_1^{02} = h_1^{03} = 0,$$

$$h_2^{01} = h_2^{02} = h_2^{03} = 0, \quad (19)$$

$$h_3^{01} = h_3^{02} = h_3^{03} = 0.$$

Now, using (6) we compute the non-vanishing components of the superpotentials in the Einstein prescription and we obtain the following expressions:

$$h_0^{01} = \frac{2x}{r^2} \frac{2M}{r} \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta, \quad (20)$$

$$h_0^{02} = \frac{2y}{r^2} \frac{2M}{r} \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta, \quad (21)$$

$$h_0^{03} = \frac{2z}{r^2} \frac{2M}{r} \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta. \quad (22)$$

Combining the line element (18), the expression for the energy from (10) and the expressions (20)-(22) for the superpotentials, one obtains the energy distribution for the new charged regular black hole in the Einstein prescription:

$$E_E = M \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta. \quad (23)$$

In order to get the momentum components we use (10) and (19) and performing the calculations we find that all the momenta are zero:

$$P_x = P_y = P_z = 0. \quad (24)$$

In the left panel of Fig. 2, we plot the energy distribution in the Einstein prescription for different values of β and $\left(\frac{q}{M}\right)^2 = 0.5$.

Using the Møller prescription, which is applied in Schwarzschild coordinates $\{t, r, \theta, \varphi\}$, the only non-vanishing superpotential is given by

$$M_0^{01} = \left(\frac{2M \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta}{r^2} - \frac{2 \left(\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right)^\beta \times \exp\left(\frac{q^2}{\beta M r}\right) \times q^2}{r^3 \left(\exp\left(\frac{q^2}{\beta M r}\right) + 1 \right)} \right) r^2 \sin \theta, \quad (25)$$

while all the other components of the Møller superpotential vanish.

Applying the aforementioned result for the line element (4) and using the expression (17) for the energy, we calculate the energy distribution in the Møller prescription:

$$E_M = \left[\frac{2}{\exp\left(\frac{q^2}{\beta M r}\right) + 1} \right]^\beta \left[M - \frac{q^2 \exp\left(\frac{q^2}{\beta M r}\right)}{r \left(\exp\left(\frac{q^2}{\beta M r}\right) + 1 \right)} \right]. \quad (26)$$

Our calculations yield for all the momenta:

$$P_r = P_\theta = P_\varphi = 0. \quad (27)$$

In the right panel of Fig. 2 we plot the energy distribution in the Møller prescription for different values of β and $\left(\frac{q}{M}\right)^2 = 0.5$. Now, in Fig. 3, we compare the expressions for energy in the Einstein and Møller prescriptions for $\beta = 0.7$. Note that for large radial distances the values of energy in both prescriptions coincide.

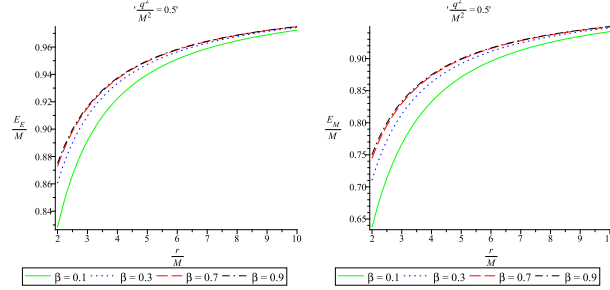


FIG. 2. (Left) Energy distribution computed by the Einstein prescription outside the outer horizon for four different values of β . (Right) Energy distribution computed by the Møller prescription outside the outer horizon for four different values of β .

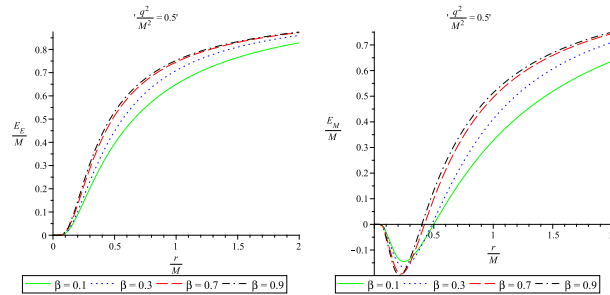


FIG. 3. Energy distribution computed by the Einstein prescription (left) and the Møller prescription (right) near the origin for different values of β and $\left(\frac{q}{M}\right)^2 = 0.5$.

V. DISCUSSION AND FINAL REMARKS

The purpose of our paper is to study the energy-momentum for a new spherically symmetric, static and charged, regular black hole solution with a nonlinear electrodynamics source using the Einstein and Møller energy-momentum

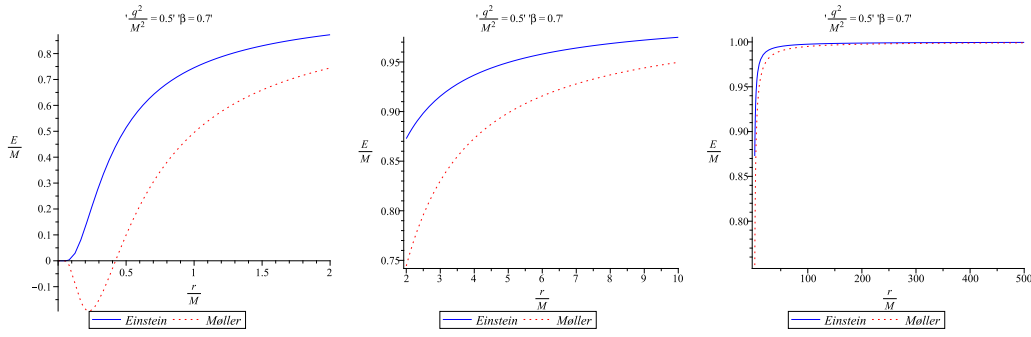


FIG. 4. For the fixed value of $\beta = 0.7$ and $(\frac{q}{M})^2 = 0.5$: (Left) Comparison of Einstein and Møller energies near the origin. (Middle) Comparison of the energy distribution performed by the Einstein and Møller prescriptions outside the outer horizon. (Right) Comparison of the energy distribution performed by the Einstein and Møller prescriptions for very large values of r .

complexes. From the calculations we conclude that in the Einstein and Møller prescriptions one obtains well-defined expressions for the energy that depend on the mass M of the black hole, its charge q , the additional factor β and on the radial coordinate r . The calculations yield that for both the aforesaid used pseudotensorial prescriptions all the momenta vanish.

Concerning the physical meaning of the energy-momentum expressions of Einstein and Møller we study the limiting behavior of the energy for $r \rightarrow \infty$, $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, and for the particular case $q = 0$. The physically meaningful results for these limiting and particular cases are presented in the following Table:

Case	$r \rightarrow \infty$	$q = 0$	$\beta \rightarrow 0$	$\beta \rightarrow \infty$
Einstein	M	M	$M \exp(-\frac{q^2}{Mr})$	$M \exp(-\frac{q^2}{2Mr})$
Møller	M	M	$[M - \frac{q^2}{r}] \exp(-\frac{q^2}{Mr})$	$[M - \frac{q^2}{2r}] \exp(-\frac{q^2}{2Mr})$

Table1

Now, some remarks are in order. Making a comparison of the results obtained for the energy distribution with the applied Einstein and Møller definitions, we conclude that for $q = 0$ and at infinity $r \rightarrow \infty$ these definitions give the same result (the ADM mass M) as that obtained for the Schwarzschild black hole solution. Moreover, this is a confirmation of Virbhadra's view point [16].

In the limiting cases $\beta \rightarrow 0$ and $\beta \rightarrow \infty$ the Einstein and Møller prescriptions provide different results. It is worth noticing that there is a difference in a factor of 2 between the exponents in the expressions for the energy. Also, we point out that although the results are different, the expressions for the energy depend on the same parameters M , q , β and r . Moreover, for $\beta \rightarrow \infty$ in both prescriptions the expressions for energy are obtained for the case of eq. (17) in [19]. The metric given by eq. (17) in [19] represents a new well-known black hole solution in the literature that contains the metric function $f(r) = 1 - \frac{2M}{r} \exp(-\frac{q^2}{2Mr})$. Interestingly, if in this case we consider $r \rightarrow \infty$ or $q = 0$ we obtain in both prescriptions of Einstein and Møller the same expression for energy which is equal to the ADM mass M . In the limiting case $\beta \rightarrow 0$ with the metric function given by $f(r) = 1 - \frac{2M}{r} \exp(-\frac{q^2}{Mr})$ we deduce, after some calculations, that the same results $E_{\text{Einstein}} = M$ and $E_{\text{Møller}} = M$ are also obtained by considering $r \rightarrow \infty$ or $q = 0$. Table 2 summarizes these results, whereby we denote E_{Einstein} by E_E and $E_{\text{Møller}}$ by E_M .

Case	$r \rightarrow \infty$	$q = 0$	$r \rightarrow \infty$	$q = 0$
$f(r) = 1 - \frac{2M}{r} \exp(-\frac{q^2}{2Mr})$	$E_E = M$	$E_E = M$	$E_M = M$	$E_M = M$
$f(r) = 1 - \frac{2M}{r} \exp(-\frac{q^2}{Mr})$	$E_E = M$	$E_E = M$	$E_M = M$	$E_M = M$

Table 2

A final remark regarding the behavior of energy as $r \rightarrow 0$ is deemed necessary. As one can see in Fig. 3, the energy obtained by the Einstein prescription tends to zero, while the energy obtained by the Møller prescription exhibits a rather strange behavior as it takes negative values in the interval $0 < r < 0.5$ for different values of β .

In the left panel of Fig. 4, the comparison of the two energies, here presented for a specific value of β , shows that the energies satisfy the inequality $E_E > E_M$ as the radial distance grows, while they tend to become equal outside the horizon for very large values of r (see Fig. 4, right panel).

The negativity of the Møller energy near the origin and, in fact, inside the inner horizon seems to be pathological and it could be attributed to the sensitivity of the Møller energy-momentum complex to the nonlinear character of the electrodynamics source. In contrast, the Einstein energy seems to be more “shielded” against this nonlinearity. We notice that a similar negativity behavior of the Møller energy has been found in [22].

In the light of the aforementioned results for the energy distribution it is obvious that the Einstein and Møller energy momentum complexes provide well-defined and physically meaningful results and are reliable prescriptions which can be used for the study of the energy momentum localization of gravitational backgrounds.

One can also ask what kind of astrophysical implications our results could have. It would be possible to investigate whether the effective gravitational mass is positive or negative by identifying the energy at radial distance r [23] with the effective gravitational mass of the astrophysical object considered inside the region determined by the distance r . But it does not seem that the present case is accessible to astrophysical observations since the negative mass region is inside the inner horizon. So the present case is interesting for positive effective gravitational mass of the astrophysical object. Moreover, we would decide whether the astrophysical object could act as a convergent or as a divergent gravitational lens [24-25].

Encouraged by these results, we plan, as a future perspective, to calculate the energy-momentum of this new charged regular black hole solution by using other energy-momentum complexes as well as the tele-parallel equivalent. These studies can further contribute to the ongoing debate on the problem of the energy-momentum localization.

ACKNOWLEDGMENTS

FR is grateful to the Inter-University Centre for Astronomy and Astrophysics (IUCAA), India for providing Associateship Programme. FR and SI are thankful to DST, Govt. of India for providing financial support under SERB and INSPIRE programme. We are thankful to the referee for his constructive suggestions.

-
- [1] L. Bel, C. R. Acad. Sci. Paris **246**, 3105 (1958); I. Robinson, Class. Quantum Grav. **14**, A331 (1997); M.A.G. Bonilla and J.M.M. Senovilla, Gen. Rel. Grav. **29**, 91 (1997); J.M.M. Senovilla, Class. Quantum Grav. **17**, 2799 (2000).
 - [2] J. D. Brown and J.W. York, Phys. Rev. **D47**, 1407 (1993); Sean A. Hayward, Phys. Rev. **D49**, 831 (1994); C. M. Chen, J. M. Nester, Class. Quantum Grav. **16**, 1279 (1999), C-C.M. Liu and S. T. Yau, Phys. Rev. Lett. **90**, 231102 (2003); L. Balart, Phys. Lett. **B687**, 280 (2010).
 - [3] A. Einstein, Preuss. Akad. Wiss. Berlin **47**, 778 (1915); Addendum-ibid. **47**, 799 (1915); A. Trautman, in *Gravitation: an Introduction to Current Research*, ed. L. Witten (Wiley, New York, 1962, p. 169).
 - [4] L. D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, 1987, p. 280).
 - [5] A. Papapetrou, Proc. R. Irish. Acad. **A52**, 11 (1948).
 - [6] P. G. Bergmann and R. Thomson, Phys. Rev. **89**, 400 (1953).
 - [7] C. Møller, Ann. Phys. (NY) **4**, 347 (1958).
 - [8] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of General Theory of Relativity* (John Wiley and Sons Inc., New York, 1972, p. 165).
 - [9] A. Qadir and M. Sharif, Phys. Lett. **A167**, 331 (1992).
 - [10] C. Møller, Nucl. Phys. **57**, 330 (1964); K. Hayashi and T. Shirafuji, Phys. Rev. **D19**, 3524 (1979); J.M. Nester, Lau Loi So and T. Vargas, Phys. Rev. **D78**, 044035 (2008); Gamal G. L. Nashed and T. Shirafuji, Int. J. Mod. Phys. **D16**, 65 (2007); J.W. Maluf, F.F. Faria and S.C. Ulhoa, Class. Quantum Grav. **24**, 2743 (2007); J.W. Maluf, M.V.O. Veiga and J.F. da Rocha-Neto, Gen. Rel. Grav. **39**, 227 (2007); A.A. Sousa, R.B. Pereira, A.C. Silva, Grav. Cosmol., **16**, 25 (2010); M. Sharif, Sumaira Taj, Astrophys. Space Sci. **325**, 75 (2010); Liu Y.X., Zhao Z.H., Yang J., and Duan Y.S., arXiv:0706.3245 [gr-qc].
 - [11] K. S. Virbhadra, Phys. Rev. **D41**, 1086 (1990); K. S. Virbhadra, Phys. Rev. **D42**, 2919 (1990); N. Rosen and K.S. Virbhadra, Gen. Rel. Grav. **25**, 429 (1993); K. S. Virbhadra and J. C. Parikh, Phys. Lett. **B331**, 302 (1994); J. M. Aguirregabiria, A. Chamorro and K. S. Virbhadra, Gen. Rel. Grav. **28**, 1393 (1996); S. S. Xulu, Mod. Phys. Lett. **A15**, 1511 (2000); S. S. Xulu, Int. J. Theor. Phys. **39**, 1153 (2000); S. S. Xulu, Int. J. Theor. Phys., **46**, 2915 (2007); P.K. Sahoo, K.L. Mahanta, D. Goit, A.K. Sihna, S. S. Xulu, U.R. Das, A. Prasad and R. Prasad, Chin. Phys. Lett. **32(2)**, 020402 (2015); S.K. Tripathy, B. Mishra, G.K. Pandey, A.K. Singh, T. Kumar, S.S. Xulu, Adv. High Energy Phys. 705262 (2015)
 - [12] I-Ching Yang and I. Radinschi, Chin. J. Phys., **42(1)**, 40 (2004); I. Radinschi, Th. Grammenos, Int. J. Theor. Phys., **47(5)**, 1363 (2008); I-Ching Yang, Chi-Long Lin, I. Radinschi, Int. J. Theor. Phys., **48(1)**, 248 (2009); I. Radinschi, F. Rahaman and A. Ghosh, Int. J. Theor. Phys. **49**, 943 (2010); I. Radinschi, F. Rahaman, A. Banerjee, Int. J. Theor. Phys., **50(9)**, 2906 (2011); I. Radinschi et al, Int.J.Theor.Phys. 51 (2012) 1425-1434 ; M. Abdel-Megied, Ragab M. Gad, Adv. High

- Energy Phys. **2010**, 379473 (2010); Ragab M. Gad, Astrophys.Space Sci. **346**, 553 (2013); T. Bringley, Mod. Phys. Lett. **A17**, 157 (2002); I. Radinschi et al, Int.J.Theor.Phys. 52 (2013) 96-104 ; M. Sukenik and J. Sima, arXiv:gr-qc/0101026; M. Sharif and Tasnim Fatima, Astrophys. Space Sci. **302**, 217 (2006); M. Sharif, M. Azam, Int. J. Mod. Phys. **A22**, 1935 (2007); P. Halpern, Astrophys. Space Sci. **306**, 279 (2006); E. C. Vagenas, Int. J. Mod. Phys. **A18**, 5781 (2003); E. C. Vagenas, Mod. Phys. Lett. **A19**, 213 (2004); E. C. Vagenas, Int. J. Mod. Phys. **D14**, 573 (2005); E. C. Vagenas, Mod. Phys. Lett. **A21**, 1947 (2006); T. Multamaki, A. Putaja, E. C. Vagenas and I. Vilja, Class. Quantum Grav. **25**, 075017 (2008); L. Balart, Mod. Phys. Lett. **A24**, 2777 (2009); Amir M. Abbassi, Saeed Mirshekari, Amir H. Abbassi, Phys. Rev. **D78**, 064053 (2008); J. Matyjasek, Mod. Phys. Lett. **A23(8)**, 591 (2008).
- [13] R. Penrose, Proc. R. Soc. London, **A381**, 53 (1982).
- [14] K.P. Tod, Proc. R. Soc. London, **A388**, 457 (1983)
- [15] L.B. Szabados, Living Rev. Relativity **12**, 4 (2009) .
- [16] K. S. Virbhadra, Phys. Rev. **D60**, 104041 (1999).
- [17] Gamal G. L. Nashed, Mod. Phys. Lett. **A22**, 1047 (2007); Gamal G. L. Nashed, Chin. Phys. Lett. **25**, 1202 (2008); Gamal G. L. Nashed, Int. J. Mod. Phys. **A23**, 1903 (2008); Gamal G.L. Nashed, Chin. Phys. **B19(2)**, 020401 (2010); Gamal G. L. Nashed, Int. J. Mod. Phys. **A25**, 2883 (2010); M. Sharif, Abdul Jawad, Astrophys. Space Sci., **331**, 257 (2011).
- [18] Chia-Chen Chang, J. M. Nester and Chiang-Mei Chen, Phys. Rev. Lett. **83**, 1897 (1999); C. M. Chen and J. M. Nester, Grav. & Cosmol., **6**, 257 (2000); Lau Loi So, J. M. Nester, Hsin Chen, in *Gravitation and Astrophysics: Proceedings of 7th Asia-Pacific Intern. Conf.*, (eds.) J.M. Nester, C-M. Chen, J-P. Hsu (World Scientific, 2007, p. 356); J. M. Nester, Chiang-Mei Chen, Jian-Liang Liu and Gang Sun, in *Relativity and Gravitation – 100 years after Einstein in Prague*, (eds.) J. Bičák and T. Ledvinka (Springer, 2014, p. 177).
- [19] L. Balart, E. C. Vagenas, Phys. Rev. **D 90**, 124045 (2014).
- [20] H. Culetu, Int. J. Theor. Phys. **54(8)**, 2855 (2015)
- [21] E. Ayón-Beato and A. García, Phys. Lett. **B 464**, 25 (1999).
- [22] I. Radinschi, F. Rahaman, Th. Grammenos, A. Spanou, and S. Islam, Adv. Math. Phys., **2015**, 530281 (2015) e-Print: arXiv:1404.6410[gr-qc] .
- [23] J. M. Cohen, F. de Felice, J. Math. Phys. **25**, 4, 992 (1984).
- [24] K.S. Virbhadra and C.R. Keeton, Phys. Rev. **D77**, 124014 (2008).
- [25] K.S. Virbhadra, Phys. Rev. **D79**, 083004 (2009).